ISRAEL JOURNAL OF MATHEMATICS, Vol. 38, No. 3, 1981

ENTROPY INCREASE IN DYNAMICAL SYSTEMS

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ABSTRACT

The rate of increase of the non-equilibrium entropy introduced by Goldstein and Penrose, defined on nonstationary probability measures for an abstract dynamical system, is quantitatively related to the Kolmogorov–Sinai entropy of the system. It is shown in particular that for ergodic systems the asymptotic rate of entropy increase coincides with the Kolmogorov–Sinai entropy.

1. Introduction

In [3] a candidate for a non-equilibrium entropy was introduced. There this quantity was shown to be non-decreasing with time, and various aspects of the entropy increase were related to familiar properties of the dynamical system. We will review these matters below. In this paper we will establish some quantative relationships between entropy increase and the Kolmogorov–Sinai (KS) entropy of the dynamical system.

Let $(\mathscr{X}, \mathscr{F}, \mu, \phi)$ be an abstract dynamical system, i.e., ϕ is an automorphism of the nonatomic Lebesgue space $(\mathscr{X}, \mathscr{F}, \mu)$ [1,8]. We denote by $H(\phi)$ the KS entropy of ϕ, and by $H(P, \phi)$ the KS entropy of ϕ relative to the finite partition P of $\mathscr{X}$. Recall that $H(\phi) = \sup_P H(P, \phi)$ and that

$$H(P, \phi) = \lim_{n \to \infty} \frac{1}{n} H\left(\bigvee_{j=0}^{n-1} \phi^j P \right);$$

here we use the notation $H(Q) \equiv -\Sigma \mu(Q_i) \log \mu(Q_i)$ for the entropy of the finite (or countable) partition $Q = (Q_i)$ of $\mathscr{X}$. (The basic definitions and properties of the KS entropy may be found in [1] or [2].)

The KS entropy is a property of the automorphism ϕ (and the "equilibrium" state μ) and not a property of non-equilibrium states (probability measures) on

[†]Supported in part by NSF Grant No. PHY 78–03816.
Received December 3, 1979

$\mathscr{X}$. Let ν be a probability measure on $\mathscr{X}$ absolutely continuous with respect to μ and let $\rho = d\nu/d\mu$. Let

$$h(\nu) = -\int \rho \log \rho d\mu, \tag{1.1}$$

and note that $-\infty \leqq h(\nu) \leqq 0$. States evolve according to

$$\nu_t = \nu \circ \phi^{-t}, \qquad t \in \mathbf{Z}, \tag{1.2}$$

and $h(\nu_t)$ is constant in t. For any sub-σ-algebra $\mathscr{A} \subset \mathscr{F}$, let the entropy of ν given $\mathscr{A}$ be

$$\begin{aligned} h(\nu, \mathscr{A}) &= -\int \rho_{\mathscr{A}} \log \rho_{\mathscr{A}} d\mu, \\ &= -\int \log \rho_{\mathscr{A}} d\nu \end{aligned} \tag{1.3}$$

where

$$\rho_{\mathscr{A}} = E(\rho \mid \mathscr{A}) \tag{1.4}$$

(the conditional expectation of ρ given $\mathscr{A}$) is the derivative of ν w.r.t. μ, both regarded as measures on $\mathscr{A}$ rather than $\mathscr{F}$. In [3] the following properties of $h(\nu, \mathscr{A})$ are established:

$$0. \qquad h(\nu_t, \mathscr{A}) = h(\nu, \phi^{-t}\mathscr{A}). \tag{1.5}$$

Suppose $\phi\mathscr{A} \supset \mathscr{A}$. Then

$$1. \qquad h(\nu_t, \mathscr{A}) \geqq h(\nu_s, \mathscr{A}) \qquad \text{for } t \geqq s. \tag{1.6}$$

$$2. \qquad \lim_{t \to \infty} h(\nu_t, \mathscr{A}) = h(\nu, \mathscr{A}_\infty), \tag{1.7}$$

where $\mathscr{A}_\infty \equiv \lim_{t \to \infty} \phi^{-t}\mathscr{A} = \bigcap_t \phi^{-t}\mathscr{A}$. In particular, if $\mathscr{A}_\infty$ is trivial, e.g. if $\mathscr{A} = \mathscr{A}_P$,[†] ϕ a K-automorphism,

$$\lim_{t \to \infty} h(\nu_t, \mathscr{A}) = h(\mu, \mathscr{A}) \, (= 0.). \tag{1.8}$$

For every sub-α-algebra satisfying $\phi\mathscr{A} \supset \mathscr{A}$ we have a (non-decreasing) *non-equilibrium entropy*, $h(\nu, \mathscr{A})$. Of particular importance to us will be the increasing σ-algebras $\mathscr{A}_P$, $\phi\mathscr{A}_P \supset \mathscr{A}$, given by

$$\mathscr{A}_P \equiv \bigvee_{n \geqq 0} \phi^{-n} P \tag{1.9}$$

[†] See equation (1.9).

for every finite partition P of $\mathscr{X}$. (It is argued in [3] that if $(\mathscr{X}, \mathscr{F}, \mu, \phi)$ represents a classical physical system, then the "physical" non-equilibrium entropy should be identified with $h(\nu, \mathscr{A}_P)$, where P is the finite partition of $\mathscr{X}$ corresponding to the outcomes of all physically possible present observations and $\mathscr{A}_P$ is thus the σ-algebra of events observable in the future.)

In this paper we shall be primarily concerned with entropy increase.

Definition 1.1. Suppose $\phi\mathscr{A} \supset \mathscr{A}$ and $\nu \ll \mu$ with $h(\nu) > -\infty$. Then

$$\Delta^{\phi}(\nu, \mathscr{A}) = h(\nu \circ \phi^{-1}, \mathscr{A}) - h(\nu, \mathscr{A}) \tag{1.10}$$

is the *entropy increase for ϕ given ν and $\mathscr{A}$*. Let

$$\Delta^{\phi}(\mathscr{A}) = \sup_{\nu} \Delta^{\phi}(\nu, \mathscr{A}) \tag{1.11}$$

be the *entropy increase for ϕ given $\mathscr{A}$*. Suppose ϕ is ergodic and $H(\phi) < \infty$. Then[†]

$$\Delta^{\phi} = \inf \Delta^{\phi}(\mathscr{A}_P), \quad P \text{ finite}, \quad H(P, \phi) = H(\phi) \tag{1.12}$$

is the *entropy increase for ϕ*, and let

$$\Delta^{\phi}_{\mathrm{as}} = \limsup_{t \to \infty} \frac{1}{t} \Delta^{\phi^t}, \qquad t \in \mathbf{Z},$$

be the *asymptotic rate of entropy increase for ϕ*.

The following properties of the entropy increase were established in [3]:

$$\text{(i) } \Delta^{\phi}(\mathscr{A}_P) > 0 \Leftrightarrow H(P, \phi) > 0. \tag{1.13}$$

(More generally, for $\mathscr{A} \subset \phi\mathscr{A}$

$$\Delta^{\phi}(\mathscr{A}) > 0 \Leftrightarrow H(\mathscr{A} \,\|\, \phi^{-1}\mathscr{A}) > 0,$$

where $H(\cdot \,\|\, \cdot)$ denotes conditional entropy [7], which will be described in section 3.) In particular,

(ii) $\Delta^{\phi}(\mathscr{A}) = 0$ for all $\mathscr{A} \subset \phi\mathscr{A} \Leftrightarrow H(\phi) = 0$,

and

[†]The infimum in (1.12) is over a nonempty set, since $H(P, \phi) = H(\phi)$ if P is a generator, $\bigvee_{n \in \mathbf{Z}} \phi^n P = \mathscr{F}$ (mod 0), and the existence of a finite generator P is assured since ϕ is ergodic and $H(\phi) < \infty$ [5, 11].

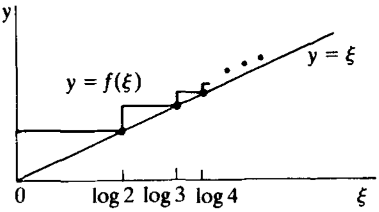

Fig. 1

(iii) $\Delta^{\phi}(\mathscr{A}_P) > 0$ for all nontrivial finite partitions $P \Leftrightarrow \phi$ is a K-automorphism.

In the next section we will describe a more detailed relationship between entropy increase and KS entropy.

2. Main results

Let f be the function graphed in Fig. 1; $f(\xi) = \log(n+1)$ for $\log n < \xi \leqq \log(n+1)$, $n = 1, 2, \cdots$; $f(0) = 0$.

THEOREM 2.1. *Suppose ϕ is ergodic and $H(\phi) < \infty$. Then*

$$\Delta^{\phi} = f(H(\phi)). \tag{2.1}$$

THEOREM 2.2. *Suppose ϕ is ergodic and $H(\phi) < \infty$. Then*

$$\Delta^{\phi}_{\mathrm{as}} = \lim_{n\to\infty} \frac{\Delta^{\phi^n}}{n} = H(\phi). \tag{2.2}$$

3. Entropy increase and information

The proof of Theorems 2.1 and 2.2 will employ the concept of "information" [7]. Let $P = (P_i)$ be a finite (or countable) partition of $\mathscr{X}$. Then $I(P)$, the information of P, is the function on $\mathscr{X}$ defined by

$$I(P)(x) = -\log \mu(P(x)), \tag{3.1}$$

where $P(x)$ is the atom of P containing the point $x \in \mathscr{X}$. (The atoms of P are $P_1, P_2, \cdots$.) Note that

$$H(P) = E(I(P)) \left(\equiv \int d\mu I(P) \right). \tag{3.2}$$

For any sub-σ-algebra $\mathscr{A} \subset \mathscr{F}$, let $\hat{\mathscr{A}}$ be the measurable partition of $\mathscr{X}$ associated with $\mathscr{A}$ [8, 1, 7], let $\mathscr{A}(x)$ be the fiber of $\hat{\mathscr{A}}$ containing x, and let $\mu(\cdot \mid \mathscr{A}(x))$ be the conditional probability given $\mathscr{A}(x)$. Since $(\mathscr{X}, \mathscr{F}, \mu)$ is a

Lebesgue space, $\mu(\cdot \mid \mathcal{A}(x))$ — a version of the conditional probability given $\mathcal{A}$ — is well defined as a measure on the Borel sets $\mathcal{F}_0$ μ a.e. Suppose $\mathcal{B}$ is another sub-σ-algebra $\subset \mathcal{F}$. Then the conditional information of $\mathcal{A}$ given $\mathcal{B}$ is the function

$$I(\mathcal{A} \| \mathcal{B})(x) = -\log \mu(\mathcal{A}(x) \mid \mathcal{B}(x)). \tag{3.3}$$

In particular, for a finite partition P,

$$I(P \| \mathcal{B})(x) = -\log \mu(P(x) \mid \mathcal{B}(x)). \tag{3.4}$$

(Since $h(\nu, \mathcal{A})$ and $I(\mathcal{A} \| \mathcal{B})$ depend, mod 0, only on the measure algebras of $\mathcal{A}$ and $\mathcal{B}$, and since for every sub-σ-algebra $\mathcal{C} \subset \mathcal{F}$ there is a sub-σ-algebra $\mathcal{C}_0 \subset \mathcal{F}_0$ (the Borel sets) which defines the same measure algebra as $\mathcal{C}$, we may, and will, assume that the sub-σ-algebras to which we refer are subalgebras of $\mathcal{F}_0$.)

$H(\mathcal{A} \| \mathcal{B})$, the conditional entropy of $\mathcal{A}$ given $\mathcal{B}$, is the expected value of the conditional information:

$$H(\mathcal{A} \| \mathcal{B}) = E(I(\mathcal{A} \| \mathcal{B})). \tag{3.5}$$

Recall that

$$\begin{aligned} H(P, \phi) &= H\Big(P \Big\| \bigvee_{n \geqq 1} \phi^{-n} P\Big) = H(P \| \phi^{-1} \mathcal{A}_P) \\ &= E(I(P \| \phi^{-1} \mathcal{A}_P)) = \int \mu(dx) H(P \mid (\phi^{-1} \mathcal{A}_P)(x)), \end{aligned} \tag{3.6}$$

where we are using the notation $H(P \mid \mathcal{B}(x))$ for the entropy of P computed using the measure $\mu(\cdot \mid \mathcal{B}(x))$ — the entropy of P relative to the fiber $\mathcal{B}(x)$. More generally,

$$H(\mathcal{A} \| \mathcal{B}) = \int \mu(dx) H(\hat{\mathcal{A}} \mid \mathcal{B}(x)), \tag{3.7}$$

where $H(\hat{\mathcal{A}} \mid \mathcal{B}(x)) = \infty$ unless $\hat{\mathcal{A}}$ is countable $\mu(\cdot \mid \mathcal{B}(x))$-mod 0.

We will also need the concept of *conditional non-equilibrium entropy*. For $\mathcal{B} \subset \mathcal{A} \subset \mathcal{F}$, let

$$\begin{aligned} h(\nu, \mathcal{A} \| \mathcal{B})(x) &= -\int \left(\frac{d\nu(\cdot \mid \mathcal{B}(x))}{d\mu(\cdot \mid \mathcal{B}(x))}\right)_{\mathcal{A}} \log \left(\frac{d\nu(\cdot \mid \mathcal{B}(x))}{d\mu(\cdot \mid \mathcal{B}(x))}\right)_{\mathcal{A}} d\mu(\cdot \mid \mathcal{B}(x)) \\ &= -\int d\nu(\cdot \mid \mathcal{B}(x)) \log \left(\frac{d\nu(\cdot \mid \mathcal{B}(x))}{d\mu(\cdot \mid \mathcal{B}(x))}\right)_{\mathcal{A}} \end{aligned} \tag{3.8}$$

where $(d\nu(\cdot\,|\,\mathscr{B}(x))/d\mu(\cdot\,|\,\mathscr{B}(x)))_{\mathscr{A}}$ is the derivative of $\nu(\cdot\,|\,\mathscr{B}(x))$ w.r.t. $\mu(\cdot\,|\,\mathscr{B}(x))$, both regarded as measures on $\mathscr{A}$. The measurability of $h(\nu,\mathscr{A}\,\|\,\mathscr{B})$ follows from the observation that

$$\left(\frac{d\nu(\cdot\,|\,\mathscr{B}(x))}{d\mu(\cdot\,|\,\mathscr{B}(x))}\right)_{\mathscr{A}}=\frac{\rho_{\mathscr{A}}}{\rho_{\mathscr{B}}}\ \left(=\frac{\rho_{\mathscr{A}}}{E(\rho_{\mathscr{A}}\,|\,\mathscr{B})}\right) \tag{3.9}$$

where $\rho = d\nu/d\mu$.

Lemma 3.1. *Suppose $\mathscr{B}\subset\mathscr{A}$ and let $\nu\ll\mu$ with $h(\nu)>-\infty$. Then*

$$h(\nu,\mathscr{B})-h(\nu,\mathscr{A})=-\int d\nu h(\nu,\mathscr{A}\,\|\,\mathscr{B}). \tag{3.10}$$

Proof.

$$h(\nu,\mathscr{B})-h(\nu,\mathscr{A})=-\int d\nu\log\rho_{\mathscr{B}}+\int d\nu\log\rho_{\mathscr{A}}$$

$$=\int d\nu\log\frac{\rho_{\mathscr{A}}}{\rho_{\mathscr{B}}}=\int\nu(dx)\int d\nu(\quad|\,\mathscr{B}(x))\log\frac{\rho_{\mathscr{A}}}{\rho_{\mathscr{B}}},$$

from which the lemma follows by (3.9).

Since $h(\nu\circ\phi^{-1},\mathscr{A})=h(\nu,\phi^{-1}\mathscr{A})$, we directly obtain from Lemma 3.1

Corollary 3.1. *Let $\mathscr{A}\subset\phi\mathscr{A}$. Then*

$$\Delta^{\phi}(\nu,\mathscr{A})=-\int d\nu h(\nu,\mathscr{A}\,\|\,\phi^{-1}\mathscr{A}). \tag{3.11}$$

Lemma 3.2. *Suppose $\mathscr{B}\subset\mathscr{A}$. Then*

$$\sup_{\substack{\nu\ll\mu\\ h(\nu)>-\infty}}(h(\nu,\mathscr{B})-h(\nu,\mathscr{A}))=\sup_x I(\mathscr{A}\,\|\,\mathscr{B})(x), \tag{3.12}$$

where the sup *on the* RHS *denotes the μ-essential supremum.*

Proof. Note that

$$\sup_{\substack{\nu\ll\mu\\ h(\nu)>-\infty}}(-h(\nu,\mathscr{A}\,\|\,\mathscr{B})(x))$$

$$=\begin{cases}-\log\mu(\mathscr{A}_{\min}(x)\,|\,\mathscr{B}(x)) & \text{if } \hat{\mathscr{A}} \text{ is finite } \mu(\cdot\,|\,\mathscr{B}(x))\text{-mod } 0,\\ \infty & \text{otherwise,}\end{cases}$$

where $\mathscr{A}_{\min}(x)$ is an atom of $\hat{\mathscr{A}}$ (w.r.t. $\mu(\cdot\,|\,\mathscr{B}(x))$) whose $\mu(\cdot\,|\,\mathscr{B}(x))$-measure is minimal. Since ν may be chosen to be more and more concentrated on fibers $\mathscr{B}(x)$ for which $-\log \mu(\mathscr{A}_{\min}(x) \mid \mathscr{B}(x))$ is "large," the lemma follows easily from Lemma 3.1. (That no measurability problems arise follows, e.g., from Rohlin [8], section 4, No. 1, applied to the quotient by $\mathscr{A}$ of the measure space $(\mathscr{X}, \mathscr{A}, \mu)$.)

From Lemma 3.2 we immediately obtain

THEOREM 3.1. *Suppose $\mathscr{A} \subset \phi\mathscr{A}$. Then*

$$\Delta^{\phi}(\mathscr{A}) = \sup I(\mathscr{A} \parallel \phi^{-1}\mathscr{A}). \tag{3.13}$$

In particular,

$$\Delta^{\phi}(\mathscr{A}_P) = \sup I(P \parallel \phi^{-1}\mathscr{A}_P). \tag{3.14}$$

4. Proof of main results

Suppose $\mathscr{A} \subset \phi\mathscr{A}$. Since, by Eq. (3.5), $H(\mathscr{A} \parallel \phi^{-1}\mathscr{A}) = E(I(\mathscr{A} \parallel \phi^{-1}\mathscr{A}))$, we may immediately obtain from Eq. (3.13) that

$$\Delta^{\phi}(\mathscr{A}) \geqq H(\mathscr{A} \parallel \phi^{-1}\mathscr{A}). \tag{4.1}$$

This can be somewhat strengthened:

THEOREM 4.1. *Suppose $\mathscr{A} \subset \phi\mathscr{A}$. Then*

$$\Delta^{\phi}(\mathscr{A}) \geqq f(H(\mathscr{A} \parallel \phi^{-1}\mathscr{A})). \tag{4.2}$$

In particular, if ϕ is ergodic and $H(\phi) < \infty$,

$$\Delta^{\phi} \geqq f(H(\phi)). \tag{4.3}$$

PROOF. To prove Eq. (4.2) it will suffice to show that

$$H(\mathscr{A} \parallel \phi^{-1}\mathscr{A}) > \log n \Rightarrow \Delta^{\phi}(\mathscr{A}) \geqq \log(n+1).$$

Suppose $H(\mathscr{A} \parallel \phi^{-1}\mathscr{A}) > \log n$. Then, by Eq. (3.7) with $\mathscr{B} = \phi^{-1}\mathscr{A}$,

$$H(\hat{\mathscr{A}} \mid (\phi^{-1}\mathscr{A})(x)) > \log n \tag{4.4}$$

for a set of $x \in \mathscr{X}$ of positive μ-measure. Since partitions P with n or fewer atoms have entropy $H(P) \leqq \log n$, we conclude that for those x for which (4.4) holds, $\hat{\mathscr{A}}$ is either infinite or finite with at least $n+1$ atoms $\mu(\quad \mid (\phi^{-1}\mathscr{A})(x))$-mod 0. If $\mathscr{A}$ has $n+1$ or more atoms $\mu(\cdot \mid (\phi^{-1}\mathscr{A})(x))$-mod 0, the measure of its smallest atom can be no greater than $1/(n+1)$. Thus, for x satisfying (4.4), $I(\mathscr{A} \parallel \phi^{-1}\mathscr{A})(y) \geqq \log(n+1)$ for a set of y of positive $\mu(dy \mid (\phi^{-1}\mathscr{A})(x))$-measure. Therefore, by Eq. (3.13), $\Delta^{\phi}(\mathscr{A}) \geqq \log(n+1)$.

Equation (4.3) now follows, since

$$\Delta^\phi(\mathscr{A}_P) \geqq f(H(\mathscr{A}_P \| \phi^{-1}\mathscr{A}_P)) = f(H(P \| \phi^{-1}\mathscr{A}_P)) = f(H(P,\phi)) = f(H(\phi))$$

if $H(P,\phi) = H(\phi)$.

PROOF OF THEOREM 2.1. We complete the proof of Theorem 2.1 by establishing, for every $\varepsilon > 0$, the existence of a finite partition P for which $H(P,\phi) = H(\phi)$ and $\Delta^\phi(\mathscr{A}_P) \leqq f(H(\phi)) + \varepsilon$. Our principal tool will be Sinai's weak isomorphism theorem, which says that every ergodic automorphism ϕ has Bernoulli factors of any entropy $\leqq H(\phi)$ [2, 10, 11]. (By Ornstein's isomorphism theorem [6], Bernoulli shifts are characterized by entropy — Bernoulli shifts of the same entropy are isomorphic.) We will use this result to establish the existence of finite partitions P such that the sequence of iterates $P, \phi P, \phi^2 P, \cdots$ has the "appropriate" structure.

Consider first the case $H(\phi) = \log n$, $n = 2, 3, \cdots$. (Theorem 2.1 is trivial in case $n = 1$.) Let $P = \{P_i\}$, $\mu(P_i) = 1/n$, be an independent generator for a Bernoulli factor of ϕ of entropy $\log n$. (That P is independent means that the (P,ϕ) process, i.e. the process $Y_n(x) = P(\phi^n x)$, is a sequence of independent random variables.) It follows from the independence of P that $I(P \| \phi^{-1}\mathscr{A}_P) = \log n$ so that we obtain using (3.14)

$$\Delta^\phi(\mathscr{A}_P) = \log n \tag{4.5}$$

for this P. Since P is a generator of a factor of entropy $\log n$, $H(P,\phi) = \log n = H(\phi)$ and Eq. (2.1) is established for this case.

Consider now the case $H(\phi) \neq \log n$, $n = 1, 2, 3, \cdots$. To establish Eq. (2.1) it will suffice to show that if $H(\phi) < \log n$, $n = 2, 3, \cdots$, then for any $\varepsilon > 0$ there exists a finite partition P such that $H(P,\phi) = H(\phi)$ and

$$\Delta^\phi(\mathscr{A}_P) \leqq \log n + \varepsilon. \tag{4.6}$$

Consider the stationary Markov chain $M_{k,l}$, $k \geqq 1$, $l \geqq 2$, with the transition structure shown in Fig. 2. All transitions are of probability $1/n$, except for those which are obviously of probability 1 and for the transitions from state 0, which are of probability $\frac{1}{2}$.

$M_{k,l}$ is obviously irreducible, and the loop at 0 makes the chain aperiodic. Thus the Markov shift $\hat{M}_{k,l}$ associated with $M_{k,l}$ — the shift on the space of two-sided trajectories equipped with the process measure — is mixing [2] and hence isomorphic to a Bernoulli shift [6].

Suppose the entropy $H_{k,l}$ of $\hat{M}_{k,l} = H(\phi)$. Then ϕ has a factor isomorphic to

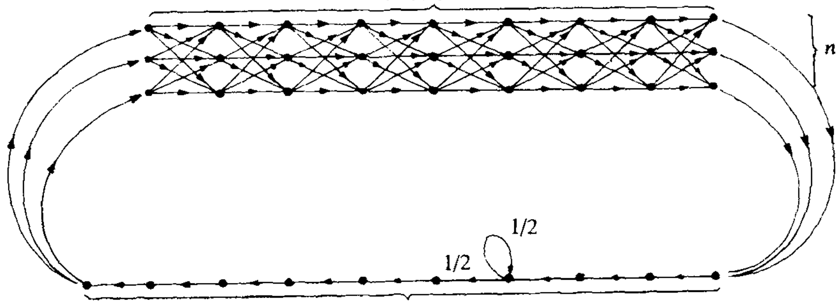

Fig. 2

$\hat{M}_{k,l}$. Let P be the natural Markov generator of this factor, so that the (P,ϕ) process is equivalent to $M_{k,l}$. Then

$$I(P \| \phi^{-1}\mathcal{A}_P) = \log n, \log 2, \text{ or } 0$$

each with positive probability, so that, using (3.14) and the Markov property, $\Delta^{\phi}(\mathcal{A}_P) = \log n$. Since $H(P,\phi) = H_{k,l} = H(\phi)$, we are done in case $H(\phi) = H_{k,l}$ for some $k \geqq 1$, $l \geqq 2$.

We now compute $H_{k,l}$. Let us denote by p_{ij} the transition probabilities and by π_i the stationary probability distribution for $M_{k,l}$. Then (using e.g. (3.6))

$$H_{k,l} = -\sum_{i,j} \pi_i p_{ij} \log p_{ij}. \tag{4.7}$$

π_i is equal to the fraction of the time spent in state i. Since the expected number of *direct* returns to state 0 (starting from state 0) is 1, $\pi = \{\pi_i\}$ has the following structure: Each state at the bottom of the structure diagram for $M_{k,l}$ and each vertical cluster of n states on top has (π) probability $1/(k+l+1)$, except for state 0 which has probability $2/(k+l+1)$. Thus

$$H_{k,l} = \frac{k}{k+l+1}\log n + \frac{2}{k+l+1}\log 2. \tag{4.8}$$

Let $D = \{H_{k,l} \mid k \geqq 1,\ l \geqq 2\}$. We are done if $H(\phi) \in D$, which is dense in $[0, \log n]$. Suppose $H(\phi) \notin D$. If $H_{k,l}$ is slightly larger than $H(\phi)$, we perturb $M_{k,l}$ slightly to obtain a (mixing) Markov shift $\hat{M}^{\delta}_{k,l}$ of entropy $H(\hat{M}^{\delta}_{k,l}) = H(\phi)$. If P is the natural Markov generator of the factor of ϕ isomorphic to $\hat{M}^{\delta}_{k,l}$, then $\Delta^{\phi}(\mathcal{A}_P)$ will be only slightly larger than $\log n$.

$M^{\delta}_{k,l}$, $0 < \delta < 1/n$, is a Markov chain with the same structure as $M_{k,l}$, except that two of the transitions from each state on top (i.e., from each state from which transitions to n other states are possible) have their probabilities changed from $1/n$, $1/n$ to $1/n+\delta$, $1/n-\delta$. The description previously given for the structure of the stationary probability distribution for $M_{k,l}$ is valid for $M^{\delta}_{k,l}$ as well. Thus

$$H(\hat{M}^{\delta}_{k,l})=\frac{k}{k+l+1}\left[\left(\frac{1}{n}+\delta\right)\log\left(\frac{n}{1+n\delta}\right)+\left(\frac{1}{n}-\delta\right)\log\left(\frac{n}{1-n\delta}\right)\right.$$

$$\left.+\left(1-\frac{2}{n}\right)\log n\right]+\frac{2}{k+l+1}\log 2 \tag{4.9}$$

$$=H_{k,l}-\frac{k}{n(k+l+1)}g(\delta),$$

where

$$g(\delta)=(1+n\delta)\log(1+n\delta)+(1-n\delta)\log(1-n\delta); \tag{4.10}$$

$g(\delta)>0$ and $g(\delta)\to 0$ as $\delta\to 0$.

Let $\bar{\delta}<1/n$ be so small that

$$\log\left(\frac{1}{n}\right)-\log\left(\frac{1}{n}-\bar{\delta}\right)<\varepsilon. \tag{4.11}$$

It follows from (4.8) and (4.9) that if $H(\phi)>0$, and if k and l are chosen so that $0<H_{k,l}-H(\phi)$ is sufficiently small, then $H(\hat{M}^{\delta}_{k,l})=H(\phi)$ for some $0<\delta<\bar{\delta}$. With this choice of δ $H(P,\phi)=H(\phi)$ and using (4.11)

$$\Delta^{\phi}(\mathscr{A}_P)=\sup I(P\,\|\,\phi^{-1}\mathscr{A}_P)<(\log n)+\varepsilon.$$

This completes the proof of Theorem 2.1. ■

Proof of Theorem 2.2. Recall [2] that

$$H(\phi^n)=nH(\phi), \tag{4.12}$$

and note that

$$\lim_{\xi\to\infty} f(\xi)/\xi=1. \tag{4.13}$$

Thus, if ϕ^n is ergodic for all $n\geqq 1$, it follows from Theorem 2.1 that

$$\frac{\Delta^{\phi^n}}{n}=\frac{1}{n}f(nH(\phi))\xrightarrow[n\to\infty]{}H(\phi).$$

At the end of section 5 we will show that if ϕ (but not necessarily ϕ^n) is ergodic, we still have that $\Delta^{\phi^n}=f(H(\phi^n))$, and the proof will be complete. ■

5. Extension to $H(\phi)=\infty$ and nonergodic ϕ

Δ^ϕ has been defined only for ϕ ergodic with $H(\phi)<\infty$. If ϕ is not ergodic, there may be no finite partition P with $H(P,\phi)=H(\phi)$, and if $H(\phi)=\infty$, then $H(P,\phi)\leqq H(P)<\infty=H(\phi)$. In either case, the infimum in (1.12) may be over the empty set.

By a theorem of Rohlin and Sinai [9], sub-σ-algebras $\mathscr{A}$ satisfying $\mathscr{A}\subset\phi\mathscr{A}$ and $H(\mathscr{A}\,\|\,\phi^{-1}\mathscr{A})=H(\phi)$ always exist, even if $H(\phi)=\infty$. Thus the quantity

$$\inf_{\substack{\mathscr{A}\subset\phi\mathscr{A}\\ H(\mathscr{A}\|\phi^{-1}\mathscr{A})=H(\phi)}}\Delta^\phi(\mathscr{A}), \tag{5.1}$$

is well defined for all ϕ. Moreover, for finite P, $H(P,\phi)=H(\phi)\Rightarrow H(\mathscr{A}_P\,\|\,\phi^{-1}\mathscr{A}_P)=H(\phi)$, so that by Eq. (4.2) the quantity (5.1) agrees with Δ^ϕ whenever the latter is defined and thus may be regarded as extending Δ^ϕ to all ϕ. Thus we will write Δ^ϕ for the quantity (5.1).

It now follows from (4.2) that Eq. (2.1) is also valid when $H(\phi)=\infty$. More generally, suppose ϕ is any (not necessarily ergodic) automorphism of $(\mathscr{X},\mathscr{F},\mu)$. Let $\mathscr{I}$ be the sub-σ-algebra of ϕ-invariant sets ($A\in\mathscr{I}\Leftrightarrow\phi A=A$). μ has the ergodic decomposition (into extremal invariant probability measures):

$$\mu(\cdot)=\int\mu(dx)\mu(\cdot\,|\,\mathscr{I}(x)).$$

For $x\in\mathscr{X}$, let ϕ_x be ϕ regarded as an automorphism of $(\mathscr{I}(x),\mu(\cdot\,|\,\mathscr{I}(x)))$. ($\phi_x$ is ergodic.) Then

THEOREM 5.1.

$$\Delta^\phi=f\Big(\sup_x H(\phi_x)\Big), \tag{5.2}$$

where the "sup" *denotes the μ-essential supremum.*

PROOF. We first prove that for $\mathscr{A}\subset\phi\mathscr{A}$ and $H(\mathscr{A}\,\|\,\phi^{-1}\mathscr{A})=H(\phi)$

$$\Delta^\phi(\mathscr{A})\geqq f\Big(\sup_x H(\phi_x)\Big). \tag{5.3}$$ ■

LEMMA 5.1. *Suppose $\mathscr{A}\subset\phi\mathscr{A}$, $H(\mathscr{A}\,\|\,\phi^{-1}\mathscr{A})=H(\phi)$, and $\mathscr{I}\subset\mathscr{A}$. Then,* mod 0,

$$E(I(\mathscr{A}\,\|\,\phi^{-1}\mathscr{A})\,|\,\mathscr{I})(x)=H(\phi_x). \tag{5.4}$$

PROOF. It follows from theorem 5.14 of [7] that for any sub-σ-algebra $\mathcal{A}$ with $\mathcal{A} \subset \phi\mathcal{A}$

$$H(\mathcal{A} \| \phi^{-1}\mathcal{A}) \leqq H(\phi). \tag{5.5}$$

Applying (5.5) to the ergodic components of ϕ gives

$$E(I(\mathcal{A} \| \phi^{-1}\mathcal{A}) | \mathcal{I})(x) \leqq H(\phi_x). \tag{5.6}$$

But

$$\begin{aligned} \int \mu(dx) E(I(\mathcal{A} \| \phi^{-1}\mathcal{A}) | \mathcal{I})(x) &= E(I(\mathcal{A} \| \phi^{-1}\mathcal{A})) \\ &= H(\mathcal{A} \| \phi^{-1}\mathcal{A}) = H(\phi) = \int \mu(dx) H(\phi_x) \qquad [4], \end{aligned} \tag{5.7}$$

and the lemma follows. ■

COROLLARY 5.1. *Suppose $\mathcal{A} \subset \phi\mathcal{A}$, $H(\mathcal{A} \| \phi^{-1}\mathcal{A}) = H(\phi)$ and $\mathcal{I} \subset \mathcal{A}$. Then,* mod 0,

$$\sup_{y \in \mathcal{I}(x)} I(\mathcal{A} \| \phi^{-1}\mathcal{A})(y) \geqq f(H(\phi_x)), \tag{5.8}$$

where "$\sup_{y \in \mathcal{I}(x)}$" *denotes the $\mu(\cdot | \mathcal{I}(x))$-essential supremum.*

PROOF. It follows from (4.2), (3.13), and (3.5) that

$$\sup I(\mathcal{A} \| \phi^{-1}\mathcal{A}) \geqq f(E(I(\mathcal{A} \| \phi^{-1}\mathcal{A}))). \tag{5.9}$$

Applying (5.9) to the ergodic components of ϕ gives

$$\sup_{y \in \mathcal{I}(x)} I(\mathcal{A} \| \phi^{-1}\mathcal{A})(y) \geqq f(E(I(\mathcal{A} \| \phi^{-1}\mathcal{A}) | \mathcal{I})(x)).$$

The corollary now follows from Lemma 5.1. ■

Equation (5.3) follows easily from (3.13) using Corollary 5.1, provided $\mathcal{I} \subset \mathcal{A}$. To handle the general case, we replace $\mathcal{A}$ by $\mathcal{A} \vee \mathcal{I}$ in (3.13). The key observation in proving that we may do so is

LEMMA 5.2. *Suppose $\mathcal{A} \subset \phi\mathcal{A}$ is strictly countably generated. Then,* mod 0,

$$\mu(\cdot | \mathcal{I}(x)) = \mu(\cdot | (\mathcal{I} \cap \mathcal{A})(x)) \tag{5.10}$$

on $\mathcal{A}$.

PROOF. Let $A \in \mathcal{A}$ and write χ_A for the characteristic function of the set A.

Then

$$\mu(A \mid \mathcal{I}(x)) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_A(\phi^j x)$$

is $\mathcal{A}$ measurable. Thus, $\mu(A \mid \mathcal{I}(x))$ is $\mathcal{I} \cap \mathcal{A}$ measurable, and therefore

$$\mu(A \mid \mathcal{I}(x)) = \mu(A \mid (\mathcal{I} \cap \mathcal{A})(x)), \bmod 0. \tag{5.11}$$

The lemma follows by letting A range over a suitable countable class. ∎

COROLLARY 5.2. *Suppose $\mathcal{A} \subset \phi\mathcal{A}$. Then*, mod 0,

$$I(\mathcal{A} \| \phi^{-1}\mathcal{A}) = I(\mathcal{A} \vee \mathcal{I} \| \phi^{-1}(\mathcal{A} \vee \mathcal{I})). \tag{5.12}$$

In particular,

$$H(\mathcal{A} \| \phi^{-1}\mathcal{A}) = H(\mathcal{A} \vee \mathcal{I} \| \phi^{-1}(\mathcal{A} \vee \mathcal{I})). \tag{5.13}$$

PROOF. If $\mathcal{A}$ were strictly countably generated, (5.12) would be an immediate consequence of Lemma 5.2. Since every sub-σ-algebra $\mathcal{A}$ of a Lebesgue space has a strictly countably generated version $\mathcal{A}_0$ [1, 8] (i.e., $\mathcal{A}_0$ and $\mathcal{A}$ define the same measure algebra), and since the information is version independent, (5.12) follows.

We remark that it follows from Corollary 5.2 that the conclusion of Lemma 5.1 is valid even without the assumption that $\mathcal{I} \subset \mathcal{A}$.

PROOF OF EQ. (5.3). Applying Corollary 5.1 to $\mathcal{A} \vee \mathcal{I}$, we obtain

$$\Delta^{\phi}(\mathcal{A}) = \sup I(\mathcal{A} \| \phi^{-1}\mathcal{A}) = \sup I(\mathcal{A} \vee \mathcal{I} \| \phi^{-1}(\mathcal{A} \vee \mathcal{I}))$$

$$= \sup_x \sup_{y \in \mathcal{I}(x)} I(\mathcal{A} \vee \mathcal{I} \| \phi^{-1}(\mathcal{A} \vee \mathcal{I})) \geqq \sup_x f(H(\phi_x)) = f\Big(\sup_x H(\phi_x)\Big). \tag{5.14}$$

The last equality follows from the fact that f is increasing and continuous from the left. ∎

To complete the proof of Theorem 5.1 we show that given $\varepsilon > 0$ there exists a sub-σ-algebra $\mathcal{A} \subset \phi\mathcal{A}$ such that

$$H(\mathcal{A} \| \phi^{-1}\mathcal{A}) = H(\phi) \tag{5.15}$$

and

$$\Delta^{\phi}(\mathcal{A}) \leqq f\Big(\sup_x H(\phi_x)\Big) + \varepsilon. \tag{5.16}$$

By working within the fibers of $\mathcal{I}$ — the ergodic components of ϕ — in the same way as in the ergodic case, we can construct a partition P such that $\mathcal{A} = \mathcal{A}_P \vee \mathcal{I}$

would satisfy (5.15) and (5.16) if P were measurable. However, it is not obvious that P can be chosen in such a way that this is so. To obtain the desired P we will use Proposition 5.1 below, a generalization of Sinai's weak isomorphism theorem.

Let T be an automorphism of the Lebesgue space $(\mathcal{Y}, \lambda)$ with finite generator $\hat{P}$. Let $\mathcal{I}$ denote the sub-σ-algebra of T-invariant sets, and let T_y, $y \in \mathcal{Y}$, be the ergodic components of T, i.e. T_y is T acting on $(\mathcal{I}(y), \lambda(\cdot \mid \mathcal{I}(y)))$. We say that $\hat{P}$ is *conditionally finitely determined* (w.r.t. T) if it is finitely determined [6] w.r.t. T_y for every $y \in \mathcal{Y}$. (Recall that finitely determined partitions $\hat{P}$ are precisely those for which the shift on the $(\hat{P}, T)$ process is isomorphic to a Bernoulli shift [6].) We write $H(T.) = H(\phi.)$ if the distribution of $H(T_y) =$ the distribution of $H(\phi_x)$. We say that T is *entropy injective* if the function $y \to H(T_y)$ is injective mod $\mathcal{I}$ (i.e., $H(T_y) = H(T_{y'}) \Rightarrow \mathcal{I}(y) = \mathcal{I}(y')$).

PROPOSITION 5.1. *Suppose* (i) $\hat{P}$ *is conditionally finitely determined w.r.t.* T, (ii) T *is entropy injective, and* (iii) $H(T.) = H(\phi.)$. *Then there exists a partition* P *of* $\mathcal{X}$ *such that* $(P, \phi) \sim (\hat{P}, T)$ (*i.e., the* (P, ϕ) *process is a copy of the* $(\hat{P}, T)$ *process in the sense that they correspond to the same measure on the space of trajectories*).

PROOF. The proposition can be proven with minor modifications of Ornstein's argument for the ergodic case (see [6], chapter 4, section 2), using versions of the ergodic theorem and the Shannon–McMillan–Breiman theorem valid for non-ergodic automorphisms [2]. We omit the details. ∎

The existence of a sub-σ-algebra $\mathcal{A} \subset \phi\mathcal{A}$ satisfying (5.15) and (5.16) may now be established as follows: We may clearly assume that

$$\sup_x H(\phi_x) = \alpha < \infty. \tag{5.17}$$

We may also assume that

$$H(\phi_x) > \beta > 0, \tag{5.18}$$

since we can always piece together the appropriate σ-algebras $\mathcal{A}_j$ on $\{H(\phi_x) \in [1/(j+1), 1/j)\}$.

Suppose $\log(n-1) < \alpha < \log n$, $n \geqq 2$. Then we may let $\mathcal{A} = \mathcal{A}_P$ (or $\mathcal{A}_P \vee \mathcal{I}$) where P is the partition (with $(P, \phi) \sim (\hat{P}, T)$) whose existence is guaranteed by Proposition 5.1 applied to the process $(\hat{P}, T)$ which we now describe.

$(\hat{P}, T)$ is a convex integral of the Markov process $M^{\delta}_{k,l}$ (see section 4):

$$(\hat{P}, T) = \int_{\mathbf{R}} \sigma(dt) M^{\delta(t)}_{k(t),l(t)}, \tag{5.19}$$

where $\sigma(dt)$, $\delta(t)$, $k(t)$, and $l(t)$ are chosen so that (ii) and (iii) of Proposition 5.1 are satisfied, $\delta(t) < \bar{\delta}$ (see Eq. (4.11)), and $k(t)$ and $l(t)$ are $\leqq N < \infty$. Since $H(\phi_x) > \beta > 0$, this can easily be done, and a finite partition $\hat{P}$ having no more than $(n+1)N$ elements thereby obtained, by using Eqs. (4.8), (4.9) and (4.10). We omit the details.

If $\alpha = \log n$, Proposition 5.1 can be applied directly to the pieces $\{H(\phi_x) \in [(\log n) - 1/j, (\log n) - 1/(j+1))\}$, for each of which $\hat{P}$ may be chosen as above, and to the piece $\{H(\phi_x) = \log n\}$, for which $(\hat{P}, T)$ may be chosen to be Bernoulli process on n equiprobable states. The proof of Theorem 5.1 is complete. ∎

COMPLETION OF PROOF OF THEOREM 2.2. We now use Theorem 5.1 to complete the proof of Theorem 2.2 (see the end of section 4).

Suppose ϕ^n is not ergodic, and let $\mathcal{I}^{(n)}$ denote the sub-σ-algebra of invariant sets for ϕ^n. Consider the action ϕ_n of ϕ w.r.t. the factor $\mathcal{I}^{(n)}$. Since ϕ_n is ergodic and ϕ_n^n is the identity transformation, the structure of ϕ_n is that of a single finite cycle. Thus $\hat{\mathcal{I}}^{(n)} = \{\hat{\mathcal{I}}_1, \cdots, \hat{\mathcal{I}}_N\}$, $N \leqq n$, is a finite partition, and $\phi\hat{\mathcal{I}}_j = \hat{\mathcal{I}}_{j+1(\mathrm{mod}\,N)}$. Thus ϕ^n acts isomorphically on each of the fibers $\mathcal{I}_j$; in particular, $H((\phi^n)_x) = H(\phi^n)$ for all $x \in \mathcal{X}$, so that by Theorem 5.1, $\Delta^{\phi^n} = f(H(\phi^n))$. ∎

6. New definition of K–S entropy

Suppose ϕ is ergodic. If we regard a vacuous infimum as infinite, Δ^{ϕ} as defined by (1.12) is well defined even for $H(\phi) = \infty$. Then the equation $H(\phi) = \Delta^{\phi}_{\mathrm{as}}$ expresses a relationship between $H(\phi)$ and Δ^{ϕ}, but since the definition of Δ^{ϕ} makes reference to $H(\phi)$, $\Delta^{\phi}_{\mathrm{as}}$, as formulated, cannot be regarded as providing an alternative definition of $H(\phi)$. However, reference to $H(\phi)$ may be avoided by referring to principal σ-algebras.

Let $\mathcal{C}$ be a sub-σ-algebra with $\phi\mathcal{C} = \mathcal{C}$. Then $\mathcal{C}$ is *principal* if $\mathcal{B} \supset \mathcal{C}$ and $\phi\mathcal{B} \supset \mathcal{B}$ implies that $\phi\mathcal{B} = \mathcal{B}$, for any $\mathcal{B} \subset \mathcal{F}$. Let $\phi_{\mathcal{C}}$ denote the factor of ϕ corresponding to $\mathcal{C}$. Then if $\mathcal{C}$ is principal, $H(\phi) = H(\phi_{\mathcal{C}})$, and conversely if $H(\phi) < \infty$, $\phi\mathcal{C} = \mathcal{C}$, and $H(\phi) = H(\phi_{\mathcal{C}})$, then $\mathcal{C}$ is principal (see [7]). Thus for P finite the condition "$H(P, \phi) = H(\phi)$" is equivalent to "$\bigvee_{n \in \mathbf{Z}} \phi^n P$ is principal." Thus Δ^{ϕ} may be defined without reference to $H(\phi)$.

ACKNOWLEDGEMENTS

The author would like to thank M. Aizenman and J. L. Lebowitz for useful discussions.

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